

GENERALIZED STATE SPACE FOR UNAWARENESS

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ABSTRACT.

Within classical epistemic logic it is easy to see that unawareness, which implies unawareness of itself is precluded. A generalized state space in connection with non-classical logic for representing non-trivial unawareness has been introduced by Heifetz, Meier and Schipper [HMS 2006] (HMS). Their approach, however, is capable of representing only true beliefs, such that erroneous beliefs count as (trivial) instances of unawareness. We generalize the HMS state space approach in a three-fold way: (i) by dropping the truth condition on beliefs, (ii) by extending it to non-classical semantics, and (iii) with respect to the information structures representable in the framework. Moreover, the HMS concept of interactive unawareness, a joint belief operator, is shown to violate basic principles of belief merging.

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1. INTRODUCTION

Dekel, Lipman and Rustichini [DLR 1998] (DLR) show that unawareness operators satisfying certain fundamental properties allow only for a trivial, or even empty, notion of unawareness in such structures. Namely, if an agent is unaware of anything, then he is unaware of everything and believes nothing. The impossibility result by DLR can be read that one is captured in the following dilemma: Either a state of unawareness is not unaware, or it is possible to believe that one is unaware of some proposition.

Several solutions to the problem have been proposed using two different syntactic structures for the objective and the subjective perspective. Modica and Rustichini [MR 1999] (MR) suggest to single out a "subjective" sub-vocabulary, describing of of which the agent is aware. Being unaware of an event is modeled by sets of states belonging to a subjective sub-space in which this event cannot be described. Halpern [Halpern 2001] introduces two different operators representing implicit and explicit beliefs¹. He proves that the MR approach can be imbedded into his formalism as a special case. Li [Li 2006] achieves the distinction by modeling information as a pair, consisting of both factual information and awareness information, which captures the agent's frame of mind. In a beautiful and appealing paper, Heifetz, Meier and Schipper (HMS) used projective state spaces to represent unawareness [HMS 2006].

All these approaches have in common that the fundamental epistemic dimension is knowledge, under which an otherwise unspecified form of *true* belief is understood. The state space models and their applications put forward nevertheless suggests that unawareness is intended a form of (higher-order) *disbelief*, which is obviously different from *disknowledge*. The former concept is of purely subjective nature, while the latter, due to the truth condition, refers to facts outside the subject. Disknowledge based on erroneous belief is perfectly compatible with awareness in the subjective sense. Assume that a subject falsely believes that p , while the contrary is true. Then the subject does not know that $\neg p$, and, since she believes that p , does not even know that she does not know, etc. Within MR and related frameworks, the subject is assigned unawareness of $\neg p$, while in the subjective sense the subject is aware of $\neg p$, since she believes that she does not believe that $\neg p$.

In the following section, we reconstruct the DLR argument in terms of epistemic logic and introduce some basic properties of Kripkean Semantics, pointing out the crucial truth axiom which has to be dropped for modelling subjective unawareness. Section 3 contains a generalisation of the HMS/MR type of state spaces, while section 4 defines the corresponding semantics and gives a sufficient condition for unawareness. Within the majority of our paper, we restrict ourselves to the case of a single person. The multi-person case of interactive unawareness, as defined by [HMS 2006], is being investigated and criticised in subsection 4.3, where we argue that either (i) common belief of all person is not included in the operator as defined there or (ii) the positive introspection property fails to hold. In section 5 demonstrate the significance of our result with an example from Principal-Agent theory. Section 6 contains an outlook to further developments and applications.

¹We prefer the notion belief over that of knowledge for the subjective concepts introduced here, since knowledge in epistemology is reserved for a special case of true, somehow justified belief.

2. UNAWARENESS

2.1. The Unawareness Paradox.

The term unawareness designates a state of higher-order disbelief. Simple disbelief could possibly be a conscious intentional state based on good reasons. Being unaware of a proposition p , on the other hand, means not only to disbelieve p , but also not to believe that p is disbelieved. The following argument shows that any concept of unawareness faces the following dilemma: either one is not unaware to be unaware that p , or it is possible to believe to be unaware of p . Both horns of the dilemma contradict the intuition that unawareness is a state free of higher-order beliefs.

Let U denote the unawareness, and B the belief operator. The first premise is that unawareness implies first and second-order disbelief,

$$(2.1) \quad Up \rightarrow \neg B\neg Bp,$$

$$(2.2) \quad Up \rightarrow \neg Bp.$$

We do not need (2.2) to derive the dilemma, but we state it here for further use. The second premise is standard for epistemic operators: Any necessary proposition is being believed,

$$(2.3) \quad (\Box p) \rightarrow Bp,$$

where \Box as usual denotes the necessity operator. Then one immediately arrives at the dilemma

$$(2.4) \quad UUp \rightarrow \Diamond BU p,$$

where, as usual, the possibility operator $\Diamond p$ stands for $\neg\Box\neg p$. For the proof assume that $\Box\neg BU p$ holds. Then by the second premise, we achieve $B\neg BU p$, and, by contraposition of the first premise with Up instantiating p , $\neg UUp$ follows. Thus $\Box\neg BU p \rightarrow \neg UUp$, or $UUp \rightarrow \neg\Box\neg BU p$.

Dropping condition (2.3), as [DLR 1998] supposed, does not make this dilemma vanishing. Surely it has been doubted whether it is adequate to presume a rational believing every theorem. But for basic theorems of the logic itself, this should nevertheless be reasonable. But even if (2.3) is weakened to hold only for some possible worlds, e.g. characterised by some precondition c , the dilemma (2.4) is still derivable under this very additional condition. But how could a possible world be constituted, where the *conceptual* paradox (2.4) is acceptable? No contingent condition could justify an analytical paradoxical result. No way out seems in sight, since dropping (2.1) means to deny unawareness at all.

It seems to be tempting to make the dilemma acceptable by interpreting \Box as provability, hoping that $UUp \rightarrow \neg BU p$ could hold without being provable. But the dilemma holds also under finite logics. Moreover, below under some reasonable assumption, $\neg BU p$ becomes provable. Our dilemma is no case for Goedel. If furthermore, \Box is being interpreted as provability with some limited proof complexity, a weakened result (with a weakened possibility operator \Diamond meaning not falsifiable with some proof complexity boundary) could be achieved. Nevertheless, also this option fails for the same reason worked out below.

The dilemma contradicts two intuitions for unawareness, which we could give the following axiomatic form. First, we expect unawareness to be a reflexive property,

such that being unaware of p implies being unaware of being unaware of p . Formally,

$$(2.5) \quad Up \rightarrow UUp.$$

Second, unawareness is supposed to be a state of higher-order disbelief, which is necessarily itself disbelieved,

$$(2.6) \quad Up \rightarrow \Box \neg BU p.$$

Here, the necessity should represent the analytical truth of the argument (if (2.6) is regarded an axiom, than the necessity operator can be omitted, since axioms are always necessary truths). Now, (2.5) together with (2.4) implies $Up \rightarrow \neg \Box \neg BU p$, thus by (2.6) we obtain $\neg Up$. Consequently, by (2.1), (2.3), (2.5), and (2.6) it follows that there cannot be a state of unawareness for *any* proposition p whatever.

2.2. Analysis.

Which axiom to drop ? (2.1) and (2.3) are out of question. To check the remaining two axioms, observe that by (2.5) and (2.2) we already achieve $Up \rightarrow \neg BU p$ as a theorem, in accordance with, although not implying, axiom (2.6). This rises suspicion that the latter might hold under suitable additional assumptions. Indeed, assume now, we interpret Up as being unaware of the *fact* that p . Thus we only consider unawareness of *true* propositions,

$$(2.7) \quad Up \rightarrow p.$$

This condition could simply be enforced if instead of U we consider an operator $\hat{U}p \equiv Up \wedge p$, defined as being unaware of a *fact*. The validity of axioms (2.1), (2.2), (2.3) is untouched, each holds for \hat{U} if and only if it holds for U . But also (2.6) could be preserved. Assume that U satisfies (2.6), then also $\hat{U}p$ does so, whenever B is a Kripke operator. For the proof let $\hat{U}p$ hold, thus Up and further $\Box \neg BU p$ by (2.6) for U . It follows $\Box ((\neg BU p) \vee (\neg Bp))$, or $\Box \neg (BU p \wedge Bp)$, which is equivalent to $\Box \neg B(U p \wedge p)$, or $\Box \neg B\hat{U}p$, for a Kripke operator B . Thus (2.6) for \hat{U} follows. Observe that a similar conclusion for condition (2.5) cannot be drawn.

Assume now, that the following intuitive principle that believed disbelief is disbelief holds,

$$(2.8) \quad B\neg Bp \rightarrow \neg Bp.$$

Then $BU p \rightarrow Bp \wedge \neg Bp$ is a theorem by (2.7) and (2.2), thus $\Box \neg BU p$, which proves (2.6). Unawareness for true propositions is therefore incompatible with axiom (2.5).

How plausible is the auxiliary assumption (2.8)? It is equivalent to

$$Bp \rightarrow \neg B\neg Bp.$$

This means that second-order disbelief is not limited to states of unawareness, but also of positive belief. But this is quite adequate, since a state where p is believed and at the same time it is believed that p is disbelieved seems schizophrenic. Thus at least for the most important case of unawareness of facts, condition (2.5) does not hold.

The four axioms (2.1), (2.3), (2.5), and (2.6) are independent: For (2.5), this has just been demonstrated. For (2.1), define Up as $p \wedge \neg Bp \wedge B\neg Bp$. Then (2.1) is false, while the other three axioms hold (since UUp is equivalent to $U(p \wedge \neg Bp \wedge B\neg Bp)$, or $Up \wedge (\neg Bp \vee \neg B\neg Bp \vee \neg BB\neg Bp) \wedge B(\neg Bp \vee \neg B\neg Bp \vee \neg BB\neg Bp)$ - the first

bracket collapses since $Up \rightarrow \neg Bp$, and the second bracket collapses if B is monotonic by $Up \rightarrow B\neg Bp$, thus $UUUp \leftrightarrow Up$ (2.5); $BUUp$ implies $Bp \wedge B\neg Bp$, which contradicts (2.8), thus (2.6) holds.) To show that (2.6) is independent of the other three axioms, consider a belief operator with $\neg Bp \leftrightarrow \neg B\neg Bp$, which must be possible at least for a given p , since (2.8) is not a theorem. Define Up as $p \wedge \neg B\neg Bp$, thus (2.1) (and even 2.2) holds. To show (2.5), observe that $UUUp$ is equivalent to $Up \wedge \neg B(\neg Bp \vee \neg B\neg B\neg Bp)$, which by $\neg Bp \leftrightarrow \neg B\neg B\neg Bp$ collapses to $Up \wedge \neg B\neg Bp$, and further to Up . Since Up is satisfiable, by the upper result, (2.6) cannot hold.

2.3. Kripke Semantics.

Both in epistemic logic and in the literature on unawareness, epistemic operators, like belief and knowledge, are expressed by possibility correspondences R , which assigns to every state w a set of states $R(w)$ the subject considers as possible. Within epistemic logic, the states are called possible worlds, and each proposition p is assigned the set $[p]$ of possible worlds, in which p holds. In a logical framework, epistemic operator are usually introduced as Kripke operators:

Definition 1. A *Kripke operator* B is defined by the truth condition

$$(2.9) \quad w \in [Bp] \Leftrightarrow R(w) \subseteq [p].$$

Any Kripke operator is characterised by the two principles

$$(2.10) \quad \begin{array}{ll} \text{Necessitation} & (\Box p) \rightarrow Bp, \\ \text{Closure} & Bp \wedge B(p \rightarrow q) \rightarrow Bq. \end{array}$$

As usual, \Box denotes necessity, which is itself a Kripke operator, here defined as $\Box p$ if and only if p holds in all possible worlds. A consequence of the two principles is the conjunction principle

$$(2.11) \quad B(p \wedge q) \leftrightarrow Bp \wedge Bq.$$

Belief and knowledge are by the majority of logic literature and all the unawareness literature considered Kripke operators, at least outside probabilistic frameworks, where the some philosophical authors see the conjunction principle violated. In all the unawareness literature one find equivalent axioms for (2.10), such as Necessitation, (2.11), and the Monotonicity axiom²

$$(2.12) \quad \text{If } \Box(p \rightarrow q), \text{ then } Bp \rightarrow Bq.$$

The conjunction of two Kripke operators with possibility correspondences R_1 and R_2 is a Kripke operator with a possibility correspondence given by $R(w) := R_1(w) \cup R_2(w)$. It is clear that the negation of a Kripke operator can not be a Kripke operator, nor will the adjunction of two Kripke operators be one in general. Symmetric operators like awareness, which have $Ap \leftrightarrow A\neg p$, can never be of the Kripke type.

The following results are standard in Kripke semantics. Several axioms correspond to properties of the defining possibility correspondence, when viewed as a

²Clearly, (2.12) follows from Closure together with Necessitation. These also imply (2.11): Since $\Box(p \wedge q \rightarrow p)$ and $\Box(p \wedge q \rightarrow q)$ holds, one concludes that $B(p \wedge q)$ implies $Bp \wedge Bq$. The converse direction follows from $\Box(p \rightarrow (q \rightarrow p \wedge q))$ by two-fold application of closure.

In the presence of Necessitation, (2.12) and (2.11) imply Closure: From Bp and $B(p \rightarrow q)$ by (2.11) we obtain $B(p \wedge (p \rightarrow q))$, and with $\Box(p \wedge (p \rightarrow q) \rightarrow q)$ from (2.12) it follows Bq .

relation on the state set with wRv written for $v \in R(w)$. The product of two relations R and Q is defined by $wRQv \leftrightarrow \exists u : wRu \wedge uQv$; I denotes the identity.

Properties of B	Axiom	Properties of R	Formula
Consistency	$Bp \rightarrow \neg B\neg p$	serial	$R(w) \neq \emptyset$
Positive Introspection	$Bp \rightarrow BBp$	transitive	$R^2 \subseteq R$
Truth	$Bp \rightarrow p$	reflexive	$I \subseteq R$

2.4. Two Types of Unawareness.

The two latter axioms, an axiom called Higher Order Disbelief

$$(2.13) \quad \neg Bp \wedge \neg B\neg Bp \rightarrow \neg B\neg B\neg Bp,$$

and the standard properties Necessitation (2.10.1), Conjunction (2.11), and Monotonicity (2.12) of a Kripke operator comprise the six properties of a knowledge operator K in [HMS 2006, prop. 2], while the Consistency axiom is contained in the definition of a possibility correspondence³. The crucial axiom is Truth, which is adequate for knowledge K , but not for belief B , since there can be false beliefs, but never false knowledge. Belief is more general than knowledge, since knowledge implies belief,

$$(2.14) \quad Kp \rightarrow Bp,$$

but not vice versa. Because these two concepts differ at least by the very truth axiom, the concept of unawareness is biased. [HMS 2006] *define* unawareness with respect to a *knowledge* operator as first and second order disknowledge,

$$(2.15) \quad U_{Kp} \leftrightarrow \neg Kp \wedge \neg K\neg Kp.$$

But the whole construction of their state spaces suggest a specific reading of unawareness as related to *belief*,

$$(2.16) \quad U_{Bp} \leftrightarrow \neg Bp \wedge \neg B\neg Bp.$$

[HMS 2006] consider a lattice of state spaces connected by projections from those with richer vocabulary down to the less expressive spaces (see section 3.2). The basic idea is that unawareness occurs in a given state for those propositions, which can not be expressed in the sublanguage associated with the space, to which the state belongs. An epistemic subject is considered unaware of a proposition p , if her epistemic state does not allow for a representation of p in the vocabulary associated to the state space. In a state of unawareness, the subject does not even think of a proposition, which is different from considerably disbelieving both the proposition and its negation, as to believe not to believe. An epistemic state of a subject can be veridic (containing only true propositions), but it need not. There could be false beliefs in the subject's epistemic state, but they do definitely not account for cases of unawareness. Therefore, we have to adhere to the second definition of unawareness relating to belief instead of knowledge.

Under certain conditions, false beliefs can be instances of unawareness in the sense of [HMS 2006] (2.15), while they are never instances of unawareness in our

³The Positive Introspection Axiom in [HMS 2006] takes the form $BBp \leftrightarrow Bp$, which is characterized by $R^2 = R$. This follows from our version of Positive Introspection and Truth.

sense (2.16). Assume that for a given proposition p , the following introspection rule holds:

$$(2.17) \quad Bp \rightarrow BKp.$$

Even if this rule is not universally valid, it can be justified under certain circumstances. It holds, for example, if the truth condition is the only difference between belief and knowledge, such that $Kp \leftrightarrow p \wedge Bp$. Then it follows that

$$\neg p \wedge Bp \rightarrow U_K p.$$

From $\neg p$, by the truth condition for knowledge we obtain the first part $\neg Kp$. From Bp , by (2.17) we obtain BKp , and further, by consistency, $\neg B\neg Kp$. Then (2.14) implies $\neg K\neg Kp$, which together with $\neg Kp$ results in $U_K p$ by definition.

[HMS 2006] derive the following properties for their unawareness operator $U = U_K$ and $B = K$ ($Ap := \neg U p$ denotes the awareness operator):

$$(2.18) \quad \begin{array}{ll} (1) & KU \text{ introspection} & \neg BUp \\ (2) & AU \text{ Introspection} & Up \leftrightarrow UUp \\ (4) & \text{Strong Plausibility} & [Up] = \bigcap_{n \in \mathbb{N}} [(\neg B)^n p] \\ (5) & \text{Weak Negative Introspection} & \neg Bp \wedge A\neg Bp \leftrightarrow B\neg Bp \\ (6) & \text{Symmetry} & Ap \leftrightarrow A\neg p \\ (7) & A\text{-Conjunction} & A(p \wedge q) \leftrightarrow Ap \wedge Aq \\ (8) & AK\text{-Self Reflection} & ABp \leftrightarrow Ap \\ (9) & AA\text{-Self Reflection} & AAp \leftrightarrow Ap \\ (10) & A\text{-Introspection} & BAp \leftrightarrow Ap \end{array}$$

Unfortunately, the proofs of [HMS 2006] present for the properties (1), (2), (4), (5) and (8) rely essentially on the Truth axiom for knowledge, and can not be carried over to our general framework, where this axiom is dropped.⁴ The proofs of properties (6), (7), (9), (10) use a here yet unexpressible property (3), which depends on structural assumptions on the state spaces we aim to generalize.

3. GENERALIZED STATE SPACES

3.1. Local State Spaces.

The idea of using non-partitional information structure to express unawareness goes back to [Geanakoplos 1989]. It was, however, Modica and Rustichini [MR 1999] who first noted that a framework connecting both a richer and a reduced state space on a joint sum space can have a globally non-partitional information structure, which is partitional on each of the two subspaces. A solution to the unawareness paradox has been offered by [HMS 2006]. Within a non-classical logic, where the tertium non datur has been dropped, they have developed a state-space model which satisfies all desiderata proposed this far in the literature, including the critical axioms (2.1), (2.3), (2.5), and (2.6). The logic is declared over its semantic declared over a state space, here interpreted as a set of possible worlds. This logic has the remarkable property that there exists a partition of the state space, such that the logic is classical on every element of the partition (proposition 2).

⁴Property (8) is elementary by definition and a property of the form $BBp \rightarrow Bp$, which follows from Truth, but is included in the [HMS 2006] version of Positive Introspection.

Definition 2. A triple $\langle \Omega, \mathcal{S}, \mathcal{A} \rangle$ is called a **local state space** if and only if \mathcal{S} is a partition of Ω called the set of **base spaces**, and \mathcal{A} is a collection of subsets of Ω satisfying

- (i) $\emptyset \in \mathcal{A}$,
 - (ii) $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcap_{i \in \mathbb{N}} A_i \in \mathcal{A}$,
 - (iii) For each $S \in \mathcal{S}$ and $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$, if $A_i \cap S \neq \emptyset$ for all $i \in \mathbb{N}$ and $\bigcap_{i \in \mathbb{N}} A_i \neq \emptyset$, then $\bigcap_{i \in \mathbb{N}} A_i \cap S \neq \emptyset$,
 - (iv) For any $A \in \mathcal{A}$, if $S \setminus A \neq \emptyset$ and $S \cap A \neq \emptyset$ for one $S \in \mathcal{S}$, then $S \setminus A \neq \emptyset$ for all $S \in \mathcal{S}$ with $S \cap A \neq \emptyset$,
 - (v) $A \in \mathcal{A} \Rightarrow A^c, A \cup A^c \in \mathcal{A}$,
- where the complement A^c is declared as

$$A^c := \bigcup \{S \setminus A \mid S \in \mathcal{S}, S \cap A \neq \emptyset\}.$$

The union of an empty set of sets by convention is set to \emptyset . Condition (i), (ii), and (v) are the closedness of \mathcal{A} under a local complement relative to the base spaced, and under countable intersection. This ensures a partial algebra to be definable with negation $\neg A := A^c$ for $A \neq \emptyset$, and conjunction declared simply by the intersection $A \wedge B := A \cap B$. The adjunction of A and B with $A^c \cap B^c \neq \emptyset$ could be defined over the de Morgan rule by $A \vee B := (A^c \cap B^c)^c$, and by $A \vee \neg A := A \cup A^c$ for direct complements. For arbitrary two elements, their union is not included, nor is an adjunction definable. Observe that we also do not request $\Omega \in \mathcal{A}$. This reflects the partiality of the algebraic structure where, in general, we have ambiguous tautological propositions $A \vee \neg A \leftrightarrow B \vee \neg B$. Condition (iii) states that a non-empty countable conjunction of propositions is consistently defined on each base space they commonly intersect. Condition (iv) ensures that a proposition which is contingent on one base space, must be contingent on each base space it is consistent with. The complementation is self-adverse except for sets with empty complement. In general, the complementation does not satisfy the de Morgan law with respect to set union and intersection. There is, however, the important exception if the two sets intersect the same base spaces.

Definition 3. Two sets $A, B \in \mathcal{A}$ are called **co-measurable** w.r.t. \mathcal{S} , iff for all $S \in \mathcal{S}$, $A \cap S \neq \emptyset \Leftrightarrow B \cap S \neq \emptyset$.

Proposition 1. (i) For $A \in \mathcal{A}$ with $A^c \neq \emptyset$, $A^{cc} = A$.
(ii) For two co-measurable sets $A, B \in \mathcal{A}$ with $A^c \cap B^c \neq \emptyset$ we have

$$(A^c \cap B^c)^c = A \cup B.$$

(iii) For any $A, B \in \mathcal{A}$ and any $S \in \mathcal{S}$ we have

$$\emptyset \neq S \cap A \subseteq S \cap B \neq S \Rightarrow A \subseteq B,$$

where the last condition $S \cap B \neq S$ can be relaxed if A and B are co-measurable.

Proof. (i)

$$\begin{aligned}
A^{cc} &= \bigcup \{S \setminus A^c \mid S \in \mathcal{S}, S \cap A^c \neq \emptyset\} \\
&= \bigcup \left\{ \bigcap \{S \setminus (T \setminus A) \mid T \in \mathcal{S}, T \cap A \neq \emptyset\} \mid S \in \mathcal{S}, S \cap A^c \neq \emptyset \right\} \\
&= \bigcup \{S \cap A \mid S \in \mathcal{S}, S \cap A \neq \emptyset, S \cap A^c \neq \emptyset\} \\
&= \bigcup \{S \cap A \mid S \in \mathcal{S}\} = A.
\end{aligned}$$

(ii) Since \mathcal{S} is a partition, two sets are identical iff they coincide on every base space $S \in \mathcal{S}$. Thus we have to show that for every $S \in \mathcal{S}$, $(A^c \cap B^c)^c \cap S = (A \cup B) \cap S$. We find

$$(A^c \cap B^c)^c \cap S = S \setminus (A^c \cap B^c) = S \setminus A^c \cup S \setminus B^c = S \setminus S \setminus A \cup S \setminus S \setminus B = (A \cup B) \cap S.$$

(iii) Let there be $A, B \in \mathcal{A}$ and $S \in \mathcal{S}$ with $\emptyset \neq S \cap A \subseteq S \cap B \neq S$. We first consider the case that $S \cap B \neq S$, or $S \setminus B \neq \emptyset$. In other words, $S \cap B^c \neq \emptyset$ with $B^c \in \mathcal{A}$ by condition (v). Since also $S \cap A \neq \emptyset$, if there was $B^c \cap A \neq \emptyset$, then, by condition (iii) we would find $S \cap (B^c \cap A) \neq \emptyset$, and further $(S \setminus B) \cap (S \cap A) \neq \emptyset$, a contradiction to $S \cap A \subseteq S \cap B$. We conclude $B^c \cap A = \emptyset$, or $A \subseteq B$.

Now to the case that $S \cap B = S$ and A, B are co-measurable. From $S \subseteq B$ and the premise $S \cap B \neq \emptyset$, we conclude from condition (iv) that for any $T \in \mathcal{S}$ with $T \cap B \neq \emptyset$, we find $T \subseteq B$. Since A and B are co-measurable, we have for any $T \in \mathcal{S}$ with $T \cap A \neq \emptyset$ also $T \cap B \neq \emptyset$, and further $T \cap A \subseteq T \subseteq B$. Since \mathcal{S} is a partition, it follows that $A \subseteq B$. ■

Definition 4. A proposition $A \in \mathcal{A}$ is **expressible** in a base space $S \in \mathcal{S}$, if and only if $S \cap A \neq \emptyset$ and $S \setminus A \neq \emptyset$. We say that a proposition $A \in \mathcal{A}$ is **non-degenerate** iff there is a base space $S \in \mathcal{S}$, in which it is expressible.

Clearly, by definition of the complement, a non-degenerate proposition $A \in \mathcal{A}$ is expressible in $S \in \mathcal{S}$, if and only if the complement A^c is. By condition (iv) of the definition of a local state space, a non-degenerated proposition is expressible in a base space $S \in \mathcal{S}$, if and only if $S \cap A \neq \emptyset$.

Definition 5. A base space $S' \in \mathcal{A}$ is called **at least as expressive** than $S \in \mathcal{A}$, denoted by $S \trianglelefteq S'$, iff each proposition from \mathcal{A} , which is consistent with (tautological on) S , is also consistent with (tautological on) S' :

$$S \trianglelefteq S' \Leftrightarrow \forall A \in \mathcal{A} : (A \cap S \neq \emptyset \Rightarrow A \cap S' \neq \emptyset) \wedge (S \subseteq A \Rightarrow S' \subseteq A).$$

By S^\uparrow we denote the **extension** of base space S to at least as expressive state spaces,

$$S^\uparrow := \bigcup \{T \in \mathcal{S} \mid S \trianglelefteq T\}.$$

The following proposition shows how the property of a local subspace inherits to substructures.

Proposition 2. Let $\langle \Omega, \mathcal{S}, \mathcal{A} \rangle$ be a local state space, and $S \in \mathcal{S}$ be a base space.

(a) The **trace space** declared by

$$\mathcal{A}_S^{tr} := \{A \cap S \mid A \in \mathcal{A}\} \cup \{S\}$$

is a σ -algebra over S .

(b) The **subspace**

$$\mathcal{A}_S^{sub} := \{A \in \mathcal{A} \mid A \cap S \neq \emptyset\} \cup \{\emptyset\}$$

forms a local state space.

(c) The collection of **final events**

$$\mathcal{A}_S^{\text{fin}} := \{A \in \mathcal{A} \mid A \cap S \neq \emptyset \wedge A \subseteq S^\uparrow\} \cup \{\emptyset, S^\uparrow\},$$

is a σ -algebra over S^\uparrow . All non-empty elements of $\mathcal{A}_S^{\text{fin}}$ are co-measurable.

3.2. Projective State Spaces.

The type of local state spaces considered by HMS [HMS 2006] is constructed by a set of disjoint spaces connected by projections. Let \mathcal{S} be a collection disjoint state spaces, each provided with a σ -algebra \mathcal{A}_S , $S \in \mathcal{S}$ ([HMS 2006] considers only the case $\mathcal{A}_S = \wp(S)$). Call $\Omega := \bigcup \mathcal{S}$ their union. The state spaces are ordered by a complete lattice \preceq , where $S \preceq T$ indicates that the vocabulary of S is contained in that of T , such that T is at least as expressive as S . For each $S, T \in \mathcal{S}$ with $S \preceq T$, there is a surjective projection $r_S^T : T \rightarrow S$, where r_S^S is the identity, and for $R \preceq S \preceq T$, the projections satisfy the commutation rule: $r_R^T = r_R^S \circ r_S^T$. We assume each projector r_S^T to be an $\langle \mathcal{A}_T, \mathcal{A}_S \rangle$ -measurable function. For any set $S \in \mathcal{S}$ and each set $B \in \mathcal{A}_S$ let

$$B^\uparrow := \bigcup_{T \succeq S} (r_S^T)^{-1}(B)$$

be called the **extension** of B . Set of this form are called **events** by [HMS 2006]. We call such an ordered set of state spaces equipped with suitable projections a **projective state space**.

Proposition 3. *For each projective state space, the set of events*

$$\mathcal{A} := \{B^\uparrow \mid B \in \mathcal{A}_S, S \in \mathcal{S}\}$$

forms a local state space $\langle \Omega, \mathcal{S}, \mathcal{A} \rangle$.

Our notion of local state spaces is a proper generalization of projective spaces. Although proposition 2a) states that local σ -algebras can be constructed from any local state space, there are ample immediate counterexamples, such as the trivial space $\mathcal{A} = \{\emptyset\}$, or the direct sum of two algebras $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, $\mathcal{A}_i \subseteq \wp(S_i)$, on two disjoint base spaces $S_1 \cap S_2 = \emptyset$, which cannot be written as a projective space. This even holds under the special conditions $\mathcal{A}_i = \wp(S_i)$ presupposed by [HMS 2006]. Even our definition of a projective state space is a useful generalisation to represent a direct sum of different versions of the same state space, which differs only in their information structure \mathcal{A}_S . We will exploit that formalism in the next section.

4. STATE SPACE SEMANTICS

4.1. State Space Models.

We restrict ourselves to a simple propositional languages L , that is, a formal language consisting only of basic propositions connected by logical junctors. Since in classical logic, all junctors could be defined by conjunction and negation, we take those two junctors as the only logical primitives. We start with a state space semantics for a classical, two-valued logic and show to extend the classical models to a three-valued framework. The third value of proposition beside true and false could be understood as expressing that a proposition is undefined, neither true nor

false, on a certain subspace. It is needed to define unawareness in a sense that overcomes the aforementioned paradox.

Definition 6. Let L be a propositional language and $\langle \Omega, \mathcal{S}, \mathcal{A} \rangle$ be a local state space. A quadruple $M = \langle [.] , \Omega, \mathcal{S}, \mathcal{A} \rangle$ with a mapping $[.]$ from statements of L to \mathcal{A} is called a **state space model** of L iff

- (i) $[p] = \Omega$, only if $\vdash p$,
- (ii) If $\vdash (p \leftrightarrow q)$, then $[p] = [q]$,
- (iii) $[p \wedge q] = [p] \cap [q]$,
- (iv) $[p] \neq \emptyset \Rightarrow [\neg p] = [p]^c$.

Thus adjunctions are evaluated according to the de Morgan law,

$$[p], [q], [p]^c \cap [q]^c \neq \emptyset \Rightarrow [p \vee q] = ([p]^c \cap [q]^c)^c.$$

Since $\neg\neg p \equiv p$, it follows that

$$[p] = \emptyset \Rightarrow [\neg p]^c = \emptyset.$$

The only restriction we impose so far on belief operators is that their defining possibility correspondence R obeys the condition that for each state w , $R(w)$ is expressible in the base space $S \ni w$ belonging to w ,

Definition 7. A **possibility correspondence** aligned to a local state space $\langle \Omega, \mathcal{S}, \mathcal{A} \rangle$ is a mapping R from Ω to non-empty elements of \mathcal{A} , such that for all $w \in \Omega$ and $S \in \mathcal{S}$

$$w \in S \Rightarrow R(w) \cap S \neq \emptyset.$$

Let B be a Kripke operator representing belief, which is defined from an aligned possibility correspondence by the truth condition (2.9). The following proposition shows that indeed for any non-degenerate proposition p , the statements Bp , $B\neg p$, $B\neg Bp$, $B\neg B\neg p$, $B\neg B\neg Bp$, $B\neg B\neg B\neg p$, ... are all false in states on base spaces, on which p is not expressible. This is in accordance with the aforementioned view on unawareness given by (2.18.4), although the positive form that the negations of those statements is true does not follow, and can not follow, as we have seen, within classical two-valued logic.

Proposition 4. If a non-degenerate proposition $[p] \in \mathcal{A}$ is not expressible in a base space $S \in \mathcal{S}$, then for all $n \geq 0$

$$[B(\neg B)^n p] \cap S = [B(\neg B)^n \neg p] \cap S = \emptyset.$$

4.2. Non-Classical Semantics.

In order to circumvene the unawareness paradox, we now introduce a semantics for state space models, which differs from the mapping $[.]$. The authors of [HMS 2006] have observed that elements S from \mathcal{S} or unions thereof have empty complement, $S^c = \emptyset$, and therefore $S^{cc} \neq S$. This led them to the conclusion that empty sets have to be individuated relative to a base set $S \in \mathcal{S}$ and indexed in the form \emptyset^S . A further definition that $(\emptyset^S)^c := S$ restitutes the double complementation rule, but, in theory, further problems could occur. If there were disjoint sets $A, B \subseteq S$, we should assign $A \cap B = \emptyset^S$. Fortunately, for the type of sets [HMS 2006] focuses on, this case is excluded. For sake of generality, Halpern and Rêgo [HR 2005] have suggested to provide an interpretation of propositions in terms

of pairs of sets instead of single sets, designating extension and anti-extension of the proposition. As a first step, we shall define an abstract algebra on suitable pairs of sets.

Definition 8. Let $\langle \Omega, \mathcal{S}, \mathcal{A} \rangle$ be a local state space. By \mathcal{A}^* we denote the smallest subset of $\mathcal{A} \times \mathcal{A}$, which is closed under the following operations:

- (i) $A \in \mathcal{A}, A \neq \emptyset \Rightarrow (A, A^c) \in \mathcal{A}^*$,
- (ii) $(A, B), (C, D) \in \mathcal{A}^* \Rightarrow (A, B) \sqcap (C, D) \in \mathcal{A}^*$,
- (iii) $(A, B) \in \mathcal{A}^* \Rightarrow \sim (A, B) \in \mathcal{A}^*$,

with

$$(A, B) \sqcap (C, D) := (A \cap C, (A \cap D) \cup (B \cap C) \cup (B \cap D)),$$

$$\sim (A, B) := (B, A)$$

The semantics we define now assigns a paraconsistent logic to language L . That means, substitution of equivalent statements are allowed only for consistent propositions, and here even not for tautologies. We regard $p \vee \neg p$ and $q \vee \neg q$ as essentially different propositions, if p and q are not co-measurable. Substitutions of junctors by equivalent expressions, however, is allowed in any context. We take L to consist only of basal propositions connected by \wedge and \neg , and take other junctors to be abbreviations of their equivalent formulation in classical logic.

Proposition 5. Let $M = \langle \cdot, \Omega, \mathcal{S}, \mathcal{A} \rangle$ be a state space model of some language L . There is a unique mapping $[\cdot]_M$ from L to \mathcal{A}^* satisfying

$$(4.1) \quad \text{If } p, \neg p \not\perp \text{ then } [p]_M = ([p], [p]^c),$$

$$(4.2) \quad \text{If } p, \neg p \not\perp \text{ and } \vdash (p \leftrightarrow q), \text{ then } [p]_M = [q]_M,$$

$$(4.3) \quad [p \wedge q]_M = [p]_M \sqcap [q]_M,$$

$$(4.4) \quad [\neg p]_M = \sim [p]_M.$$

We call $[\cdot]_M$ the **interpretation function** of state space model M and write $\models_M p$ for $[p]_M = (\Omega, \emptyset)$.

4.3. Interactive Unawareness.

Belief merging is about mapping a set of individual belief operators B_1, \dots, B_n of n persons to an aggregated belief operator B , which represents the belief of the group. If the belief structure is capable of representing unawareness, then, following [HMS 2006], we speak of interactive unawareness. There are several plausible axiom one can propose to such an operator. First of all, common belief should also be group belief,

$$(CB) \quad B_1 p \wedge \dots \wedge B_n p \rightarrow B p.$$

Secondly, also common unawareness should be group unawareness,

$$(CU) \quad U_1 p \wedge \dots \wedge U_n p \rightarrow U p.$$

Condition (CB) could be easily rewritten in terms of the generating possibility correspondences R_1, \dots, R_n and R of B_1, \dots, B_n and B , respectively:

$$(4.5) \quad R(w) \subseteq \bigcup_{i=1}^n R_i(w).$$

The common knowledge operator in [HMS 2006] is defined in two steps. First, the "everybody knows" operator is defined which represents the common belief,

$$(4.6) \quad \bar{B}p \leftrightarrow B_1p \wedge \dots \wedge B_np.$$

Second, the common knowledge operator is formed by iteration to ensure the condition of positive introspection,

$$(4.7) \quad Bp \leftrightarrow \bar{B}p \wedge \bar{B}\bar{B}p \wedge \dots$$

Definition (4.6) can be expressed by the possibility correspondence \bar{R} of \bar{B} ,

$$(4.8) \quad \bar{R}(w) = \bigcup_{i=1}^n R_i(w).$$

This condition already assures that the possibility correspondence R is aligned to the common local state space $\langle \Omega, \mathcal{S}, \mathcal{A} \rangle$: Take arbitrary $S \in \mathcal{S}$ and $w \in S$. Then from (4.8) we obtain

$$\bar{R}(w) \cap S = \bigcup_{i=1}^n (R_i(w) \cap S) \neq \emptyset,$$

which has to be shown. The alignment property, in turn, provides a necessary condition for (CU). If, for every person i and world w , unawareness of non-degenerate proposition $[p] \in \mathcal{A}$ only occurs in the case mentioned by proposition 4, namely if the proposition is not expressible in the subspace $S \ni w$, then this very proposition implies (CU) and even its conversion. Under this additional condition, indeed both axioms (CB) and (CU) holds for the "everybody knows" operator \bar{B} .

For the common knowledge operator B , however, axiom (CB) is violated. This operator was introduced to ensure positive introspection. Definition (4.7) is equivalent to R being the transitive closure of \bar{R} . Now, since in general, \bar{R} is not transitive even if every R_i is, the case $R(w) \not\supseteq \bar{R}(w)$ can occur, such that (4.5) is violated. Nevertheless, the alignment property is satisfied, since $R(w) \cap S \supseteq \bar{R}(w) \cap S \neq \emptyset$ for $w \in S$. The failure of (CB) constitutes a major objection against the definition of interactive unawareness by [HMS 2006]. Whether there is a belief operator which satisfies both axioms (CB) and (CU) as well as positive introspection, is an open problem.

A simple example demonstrates that failure. Let $\Omega = \{u, v, w\}$, and R_1, R_2 preorders of Ω with $vR_1wR_1uR_1u$ and $uR_2wR_2vR_2v$. Then we find $\bar{R}(u) = \bar{R}(v) = \Omega$ and $\bar{R}(w) = \{u, v\}$. Thus $w\bar{R}u$ and $u\bar{R}w$, but not $w\bar{R}w$. Thus \bar{R} is intransitive, and its transitive closure R is the universal relation. Especially, $R(w) = \Omega \not\subseteq \{u, v\} = \bar{R}(w) = R_1(w) \cup R_2(w)$, in contrast to (4.5). Thus B , as declared by (4.7), violates (CB).

5. EXAMPLE

Our simple example is borrowed from the Principal-Agent model framework where the (risk-neutral) Principal faced the so-called *adverse selection problem*. We will show that being unaware or having false belief leads to different results in terms of optimal contract.

Suppose that $X = \{x_1, \dots, x_n\}$ is the set of possible monetary outcomes that can be obtained by the Principal from his relationship with the Agents. Suppose

that there are two types of Agents (the high-productivity agents and the low-productivity one) in the population and that the Principal does not observe the agents' types. Hence the agents' type is a private information, that is to say that from the point of view of the Principal, their type is a random variable. Suppose that the utility function of the high-productivity agents is $U^H(w, e) = u(w) - \theta_H v(e)$ where w is the wage, e is the agents' verifiable effort, u is a strictly increasing concave function, v is a strictly increasing convex function and θ_H is the productivity factor of the high-productivity agents. Likewise, the utility function of the low-productivity agents is $U^L(w, e) = u(w) - \theta_L v(e)$ where θ_L is the productivity factor of the low-productivity agents. Obviously, we assume that $\theta_L > \theta_H > 0$. Roughly speaking, the two type of agents differ w.r.t their disutility of effort: respectively $\theta_H v(e)$ for the high-productivity agents and $\theta_L v(e)$ for the low-productivity agents.

Let \tilde{p} be the Principal's *subjective probability* that an agent is of high-productivity type. Then it is well-known in the literature (see for instance Laffont and Martimort [LM 2002]) that the optimal contract $\{(w_H^*, e_H^*), (w_L^*, e_L^*)\}$ is the solution of the following program:

$$\underset{\{(w_H, e_H), (w_L, e_L)\}}{\text{Max}} \quad \tilde{p}[B(e_H) - w_H] + (1 - \tilde{p})[B(e_L) - w_L]$$

under the constraints:

$$\begin{aligned} u(w_H) - \theta_H v(e_H) &\geq \underline{U} \\ u(w_L) - \theta_L v(e_L) &\geq \underline{U} \\ u(w_H) - \theta_H v(e_H) &\geq u(w_L) - \theta_H v(e_L) \\ u(w_L) - \theta_L v(e_L) &\geq u(w_H) - \theta_L v(e_H) \end{aligned}$$

where $B(e_H) = \sum_{i=1}^n Pr(x = x_i | e = e_H) x_i$ is the strictly increasing concave expected profit conditional to the fact that the agent is a high-productivity agent. Likewise $B(e_L)$ is the strictly increasing concave expected profit conditional to the fact that the agent is a low-productivity agent.

The first two constraints are called *participation constraints* since they require the optimal contract to be such that the agent gets at least the reservation utility \underline{U} which is the weakest utility he can obtain elsewhere. The third and the fourth constraints are called *incentive constraints* since they prevent the agent to lie. For instance the optimal contract $\{(w_H^*, e_H^*), (w_L^*, e_L^*)\}$ is such that only the low-productivity agents will choose the contract (w_L^*, e_L^*) and only the high-productivity agents will choose the contract (w_H^*, e_H^*) .

The optimal contract $\{(w_H^*, e_H^*), (w_L^*, e_L^*)\}$ is such that:

$$(5.1) \quad u(w_H) - \theta_H v(e_H) = \underline{U} + \underbrace{(\theta_L - \theta_H)v(e_L)}_{\text{Informational Rent for H-type agents}}$$

$$(5.2) \quad u(w_L) - \theta_L v(e_L) = \underline{U}$$

$$(5.3) \quad B'(e_H) = \frac{v'(e_H)}{u'(w_H)}$$

$$(5.4) \quad B'(e_L) = \frac{\theta_L v'(e_L)}{u'(w_L)} + \underbrace{\frac{\tilde{p}(\theta_L - \theta_H)v'(e_L)}{(1 - \tilde{p})u'(w_H)}}_{\text{Distortion for L-type agents}}$$

Obviously the Principal's subjective probability \tilde{p} plays a crucial role here. Indeed the smaller this probability the smaller the distortion $\frac{\tilde{p}(\theta_L - \theta_H)v'(e_L)}{(1 - \tilde{p})u'(w_H)}$ and the higher the *informational rent* $(\theta_L - \theta_H)v(e_L)$, for the High-productivity individuals: when the Principal thinks that the proportion of High-productivity individuals is small then it is better for him to reduce the distortion for the Low-productivity individuals and to increase the informational rent to the High-productivity individuals (since they are scarce). Conversely the higher the probability \tilde{p} the higher the distortion $\frac{\tilde{p}(\theta_L - \theta_H)v'(e_L)}{(1 - \tilde{p})u'(w_H)}$ and the smaller the informational rent for the High-productivity individuals: when the Principal thinks that the proportion of High-productivity individuals is high then it is better for him to reduce the informational rent to the H-type individuals and to increase the distortion for the Low-productivity individuals (since they are scarce).

However the type of the individuals (low or high productivity) will be correlated to their socio-demographic characteristics. For instance, the likelihood to be a high-productivity individual can be increased w.r.t the diploma.

Suppose now that the Principal can, for free, observe for each individual j , the set $\mathcal{C}_j = \{c_{1j}, c_{2j}, \dots, c_{kj}\}$ of k socio-demographic characteristics (for instance, age, diploma, gender, race,...) which he *believes* to be positively correlated with this individual's unobserved productivity. Then it must be the case (see Blackwell theorem) that the Principal will take into account to the distribution of these characteristics in the population in order to estimate his subjective probability \tilde{p} .

Suppose also that there exists for each individual j a set $\mathcal{I}_j = \{c_{k+1j}, c_{k+2j}, \dots, c_{Kj}\}$ of $K - k$ individual characteristics which may have positive influence on this agent's productivity and about which the Principal is either *unaware* or have *false beliefs*. Such a characteristic could be unobservable like the team spirit or the self-esteem. However it could be observable like the gender, the race,...

Let us provide the following two cases (neglecting the subscript j).

- In the first case, the Principal has *true beliefs* about all the characteristics c_1, c_2, \dots, c_k (that is to say, the beliefs of the Principal concerning the correlation between these characteristics and the agents' productivity are true), however he is *unaware* about all the characteristics $c_{k+1}, c_{k+2}, \dots, c_K$. Let \tilde{p}_1 be the Principal's subjective probability that an individual is a high-productivity one.
- In the second case, the Principal has *true beliefs* about all the characteristics c_1, c_2, \dots, c_k , however he has *false beliefs*⁵ about all the characteristics $c_{k+1}, c_{k+2}, \dots, c_K$. Let \tilde{p}_2 be the Principal's subjective probability that an individual is a high-productivity one.

⁵An example of false belief is "all women are low-productivity individuals, except my wife".

The subjective probabilities \tilde{p}_1 and \tilde{p}_2 from the two cases will be different. Indeed in the first case (to the contrary to the second case), since the Principal is unaware about the characteristics $c_{k+1}, c_{k+2}, \dots, c_K$, he will not take them into account when calculating \tilde{p}_1 . Hence the optimal contracts from the two cases will be different. That is to say, being unaware or having false belief leads to different solutions in terms of optimal contract.

Let us take for instance $K = 2$ where c_1 is the diploma (High/Low) and c_2 is the gender (Men/Women). Suppose that the proportion of High Diploma individuals in the population is 20% and that the proportion of Men is 50%. Suppose also that the Principal has true belief about the diploma ("Higher Diploma individuals are more productive"). Concerning the characteristic c_2 , let us assume that the Principal is either unaware (*first case*) or (*second case*) has false belief ("All women are low-productivity individuals"). Hence in the first case, the Principal's subjective probability that an individual is a high-productivity one will be 0.2 while in the second case, his subjective probability will be 0.1. This difference will, of course, dramatically have a consequence on the optimal contract.

6. CONCLUSION

We have seen that the state space approach of HMS can be generalised to a framework avoiding the truth condition for belief. Our example demonstrates that this difference matters: false beliefs lead to different probability estimations that no beliefs at all, i.e. unawareness. Consequently, a feasible framework must be able to capture both in the same environment. Dropping the truth condition, however, has the consequence that most of the properties (2.18) HMS claims do not hold any more in general. Further axiomatic research is to reveal constraints on the possibility correspondence underlying the belief operator under which these properties hold. We are optimistic that this could be performed in more generality, since there is no reason to believe that the sentences (2.18) altogether could imply the crucial truth axiom.

There is a serious problem with the joint or interactive belief HMS are defining. Either (i) common belief of all person is not included in the operator as defined there or (ii) the positive introspection property fails to hold. It is still unknown, and an important path for future research whether a more adequate definition can be found. If the answer turns out negative, it would lead to an impossibility result for the representation of joint beliefs in generalised Kripke frameworks.

This article shows that the case of the impossibility result laid down by DLR about non-representability of (non-trivial) unawareness in a classical Kripke framework has not been closed by HMS. The HMS framework is a very powerful result, at least capable of representing individual unawareness, but it applies only to the very special case of true beliefs. There is need of further basic research in that subject.

APPENDIX A: PROOFS

Proof of proposition 2. (a) (i) Clearly, $\emptyset, S \in \mathcal{A}_S^{\text{tr}}$. (ii) For $\{A_i \cap S\}_{i \in \mathbb{N}} \subseteq \mathcal{A}_S^{\text{tr}}$, we have $\bigcap_{i \in \mathbb{N}} (A_i \cap S) = (\bigcap_{i \in \mathbb{N}} A_i) \cap S \in \mathcal{A}_S^{\text{tr}}$. (iii) For $A \cap S \in \mathcal{A}_S^{\text{tr}}$ we find $S \setminus (A \cap S) = S \setminus A = A^c \cap S \in \mathcal{A}_S^{\text{tr}}$.

(b) (i) Clearly, $\emptyset \in \mathcal{A}_S^{\text{sub}}$. (ii) Let $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}_S^{\text{sub}}$, thus $A_i \cap S \neq \emptyset$. Let $A := \bigcap_{i \in \mathbb{N}} A_i$. If $A = \emptyset$, there is nothing to be shown, so assume $A \neq \emptyset$. Then, by property (iii) of the local state spaces, $A \cap S \neq \emptyset$. Therefore, $A \in \mathcal{A}_S^{\text{sub}}$. Properties (iii) and (iv) inherit immediately. (v) If $A \in \mathcal{A}_S^{\text{sub}}$, then either $S \subseteq A$, or $S \setminus A \neq \emptyset$. In the first case, it follows from property (iv) that $T \subseteq A$ for every $T \in \mathcal{S}$ with $T \cap A \neq \emptyset$. This gives us $A^c = \emptyset \in \mathcal{A}_S^{\text{sub}}$. In the latter case, $A^c \cap S \neq \emptyset$, and thus $A^c \in \mathcal{A}_S^{\text{sub}}$. In either case, $(A \cup A^c) \cap S \neq \emptyset$, such that $A \cup A^c \in \mathcal{A}_S^{\text{sub}}$.

(c) First, we remark that by the definition of \preceq , for every $A \in \mathcal{A}_S^{\text{fin}}$,

$$S \subseteq A \Rightarrow A = S^\uparrow.$$

Second, we show that for every nonempty $A \in \mathcal{A}_S^{\text{fin}}$, $A \neq \emptyset$, A and A^c are co-measurable, and

$$A \cup A^c = S^\uparrow.$$

If $S \subseteq A$, then by the first remark, $A = S^\uparrow$, and $A^c = \emptyset$, thus the equation holds trivially. Otherwise, $S \not\subseteq A$, that is $S \setminus A \neq \emptyset$. By property (iv), for all $T \in \mathcal{S}$ with $T \cap A \neq \emptyset$, we also have $\emptyset \neq T \setminus A = T \cap A^c$. Conversely, by the definition of the complement, each $T \in \mathcal{S}$ with $T \cap A^c \neq \emptyset$ must also have $T \cap A \neq \emptyset$. We arrive at $T \cap A \neq \emptyset \Leftrightarrow T \cap A^c \neq \emptyset$, that is, A and A^c are co-measurable. Then $A \cup A^c = S^\uparrow$ by construction of $\mathcal{A}_S^{\text{fin}}$.

(i) Clearly, $\emptyset, S^\uparrow \in \mathcal{A}_S^{\text{fin}}$ by construction.

(ii) For a countable subset $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}_S^{\text{fin}}$, write $A := \bigcap_{i \in \mathbb{N}} A_i$ for their intersection. If $A = \emptyset$, then $A \in \mathcal{A}_S^{\text{fin}}$ by (i), so assume $A \neq \emptyset$. By definition of $\mathcal{A}_S^{\text{fin}}$ we have $A \subseteq S^\uparrow$. From property (iii) of the local state space it follows that $A \cap S \neq \emptyset$. Thus $A \in \mathcal{A}_S^{\text{fin}}$.

(iii) If $A = \emptyset$ then $S^\uparrow \setminus A = S^\uparrow \in \mathcal{A}_S^{\text{fin}}$, as stated by (i). If $S \subseteq A$, then by the first remark, $A = S^\uparrow$, and $S^\uparrow \setminus A = \emptyset \in \mathcal{A}_S^{\text{fin}}$. Otherwise $S \setminus A \neq \emptyset$, and we find $S^\uparrow \setminus A = A^c \in \mathcal{A}$ by the second remark. Since $S \cap (S^\uparrow \setminus A) = S \setminus A \neq \emptyset$, and $S^\uparrow \setminus A \subseteq S^\uparrow$, we conclude $S^\uparrow \setminus A \in \mathcal{A}_S^{\text{fin}}$. ■

Proof of proposition 3. (i) Clearly, \emptyset is an event, since $\emptyset = (r_S^T)^{-1}(\emptyset)$ for any S, T with $S \preceq T$.

(ii) Let $\{B_i^\uparrow\}_{i \in \mathbb{N}}$ be a sequence of event, and let $D = \bigcap_{i \in \mathbb{N}} B_i^\uparrow$ be their intersection. If $D = \emptyset$, there is nothing to be shown, so assume $D \neq \emptyset$. By definition, each event can be written in the form

$$(6.1) \quad B_i^\uparrow = \bigcup_{T \succeq S_i} (r_{S_i}^T)^{-1}(B_i)$$

with $B_i \in \mathcal{A}_{S_i}$. Now, let $\mathcal{S}_D = \{S \in \mathcal{S} \mid S \cap D \neq \emptyset\}$ be the set of state spaces intersecting with D . Since $D \neq \emptyset$, \mathcal{S}_D is non-empty. For any $S \in \mathcal{S}_D$ and $T \in \mathcal{S}$ with $S \preceq T$, we show that $T \in \mathcal{S}_D$: By construction, $B_i^\uparrow \cap S = (r_{S_i}^S)^{-1}(B_i)$, and thus for a T with $S \preceq T$, $B_i^\uparrow \cap T = (r_{S_i}^T)^{-1}(B_i)$. Letting $C_i := (r_{S_i}^S)^{-1}(B_i) = B_i^\uparrow \cap S$, then with the commutation property of the projectors we can write $B_i^\uparrow \cap T = (r_S^T)^{-1}(C_i)$.

Thus

$$(6.2) \quad D \cap T = \bigcap_{i \in \mathbb{N}} (B_i^\dagger \cap T) = (r_S^T)^{-1}(C) \text{ with } C = \bigcap_{i \in \mathbb{N}} C_i.$$

Now $C = D \cap S \neq \emptyset$ by assumption, and therefore, since the projections are surjective, $D \cap T \neq \emptyset$, or $T \in \mathcal{S}_D$.

Recall that \mathcal{S} is a complete lattice and \mathcal{S}_D an nonempty subset. Let S be the infimum of \mathcal{S}_D . We have to show that $S \in \mathcal{S}_D$. From the upper representation (6.1) of the events B_i^\dagger it follows that $S_i \preceq T$ for every $T \in \mathcal{S}_D$. By the property of the infimum we also have $S_i \preceq S \preceq T$ for every $T \in \mathcal{S}_D$ and every $i \in \mathbb{N}$. With the same argument as above we can write for an arbitrary $T \in \mathcal{S}_D$

$$(6.3) \quad D \cap T = (r_S^T)^{-1}(D \cap S) \text{ and } D \cap S = \bigcap_{i \in \mathbb{N}} (B_i^\dagger \cap S) = \bigcap_{i \in \mathbb{N}} (r_{S_i}^S)^{-1}(B_i).$$

Therefore, since $D \cap T \neq \emptyset$, also $D \cap S \neq \emptyset$, and we conclude that the infimum $S \in \mathcal{S}_D$.

Taking the last both results together we can write $\mathcal{S}_D = \{T \in \mathcal{S} \mid S \preceq T\}$. We are now in a position to show that D is an event B^\dagger with $B = D \cap S$. From the right hand side of (6.3) we see that, because $(r_{S_i}^S)^{-1}(B_i) \in \mathcal{A}_S$ and $S \in \mathcal{A}_S$, we also have $B \in \mathcal{A}_S$, since \mathcal{A}_S is a σ -algebra. It finally follows from the left hand side of (6.3) and (6.1) that $D = B^\dagger$.

(iii) This case follows as a corollary from the proof of (ii) just stated.

(iv) Assume any event $B^\dagger \in \mathcal{A}$ with $S \setminus B^\dagger \neq \emptyset$ and $S \cap B^\dagger \neq \emptyset$ for a given $S \in \mathcal{S}$. Take any $T \in \mathcal{S}$ with $T \cap B^\dagger \neq \emptyset$. We have to prove that $T \setminus B^\dagger \neq \emptyset$. Let R be the infimum of S and T w.r.t. \preceq . Then $S \cap B^\dagger = (r_R^S)^{-1}(R \cap B^\dagger)$, and $T \cap B^\dagger = (r_R^T)^{-1}(R \cap B^\dagger)$, as well as $S \setminus B^\dagger = (r_R^S)^{-1}(R \setminus B^\dagger)$ and $T \setminus B^\dagger = (r_R^T)^{-1}(R \setminus B^\dagger)$. Since $S \setminus B^\dagger \neq \emptyset$, we also find $R \setminus B^\dagger \neq \emptyset$, and conclude $T \setminus B^\dagger \neq \emptyset$, which had to be shown.

(v) It is sufficient to show that for $B \in \mathcal{A}_S$,

$$(6.4) \quad B^\dagger = \bigcup_{T \succeq S} (r_S^T)^{-1}(B) \Rightarrow (B^\dagger)^c = \bigcup_{T \succeq S} (r_S^T)^{-1}(S \setminus B),$$

then, clearly, $(B^\dagger)^c$ is an event and $B^\dagger \cup (B^\dagger)^c = S^\dagger$, too. For $T \succeq S$, by definition $(B^\dagger)^c \cap T = T \setminus B^\dagger = T \setminus (r_S^T)^{-1}(B) = (r_S^T)^{-1}(S \setminus B)$, thus equation (6.4) follows. ■

Proof of proposition 4. The proof of $[p] \cap S = \emptyset \Rightarrow [B(\neg B)^n p] \cap S = \emptyset$ is by induction. For $n = 0$ we have to show that

$$(6.5) \quad [p] \cap S = \emptyset \Rightarrow [Bp] \cap S = \emptyset.$$

Assume $[p] \cap S = \emptyset$ and $w \in [Bp] \cap S$. From the definition of belief operator we find $R(w) \subseteq [p]$, and further, by the definition of a possibility correspondence, $\emptyset \neq R(w) \cap S \subseteq [p] \cap S$, a contradiction to the assumption that $[p] \cap S = \emptyset$. Thus $[Bp] \cap S = \emptyset$.

The induction step $n \rightarrow n + 1$ is then immediate by the observation that from $[B(\neg B)^n p] \cap S = \emptyset$, by the definition of complement it follows $\emptyset = [B(\neg B)^n p]^c \cap S = [\neg B(\neg B)^n p] \cap S$, from which by (6.5) we can conclude $[B(\neg B)^{n+1} p] \cap S = \emptyset$.

Since p is non-degenerate and not expressible in S , we also have $[\neg p] \cap S = [p]^c \cap S = \emptyset$. Then $[B(\neg B)^n \neg p] \cap S = \emptyset$ follows for all $n \geq 0$ from the case just proven. ■

Proof of proposition 5. Assume first that p is a propositions, which are neither contradictory nor tautological. Then $[p]_M$ is already declared by (4.1). This definition already ensures (4.2) to hold by property (ii) of definition 6. It remains to show (4.3) and (4.4) for non-tautological and consistent propositions.

(4.3): If $p \wedge q$ is consistent, then $[p \wedge q] \neq \emptyset$. By (4.1),

$$[p \wedge q]_M = ([p \wedge q], [p \wedge q]^c) = ([p] \cap [q], ([p] \cap [q])^c).$$

We have to show that

$$([p] \cap [q])^c = ([p] \cap [q]^c) \cup ([p]^c \cap [q]) \cup ([p]^c \cap [q]^c).$$

It is sufficient to prove this equation on every base space $S \in \mathcal{S}$. Assume $[p \wedge q] \cap S \neq \emptyset$. Then the left hand side evaluates on S to

$$([p] \cap [q])^c \cap S = S \setminus ([p] \cap [q]) = S \setminus [p] \cup S \setminus [q].$$

Since also $[p] \cap S \neq \emptyset$ and $[q] \cap S \neq \emptyset$ we find for the right hand side summands

$$\begin{aligned} ([p] \cap [q]^c) \cap S &= [p] \cap S \setminus [q] \\ ([p]^c \cap [q]) \cap S &= S \setminus [p] \cap [q] \\ ([p]^c \cap [q]^c) \cap S &= S \setminus [p] \cap S \setminus [q] \end{aligned}$$

The union of these three terms is just $S \setminus [p] \cup S \setminus [q]$, what has to be shown.

(4.4): We have to show that $[\neg p]_M = \sim [p]_M$ holds for a consistent and non-tautological p . Since $[\neg p] \neq \emptyset$, we find $[\neg p]_M = ([\neg p], [\neg p]^c)$. Since also $[p] \neq \emptyset$, by property (iv) of definition 6 we obtain $[\neg p] = [p]^c$, and further $[\neg p]_M = ([p]^c, [p]^{cc})$. Since $[p]^c = [\neg p] \neq \emptyset$ we can apply proposition 1, and achieve $[\neg p]_M = ([p]^c, [p]) = \sim [p]_M$, what has to be shown.

Now let p be either be contradictory or tautological. Eliminate all logical junctors in p by logically equivalent expressions in terms of \wedge and \neg , e.g. expressions $q \vee r$ by $\neg(\neg q \wedge \neg r)$, $q \rightarrow r$ by $\neg(q \wedge \neg r)$ etc. Now p consists only of basal propositions connected by \wedge and \neg . Since basal proposition are never tautological or contradictory, their value is already defined by (4.1). Then the value of p , $[p]_M$, can be recursively obtained by (4.3) and (4.4). Thus we have shown that the interpretation function is uniquely determined. Equations (4.1) and (4.2) do not apply in case of a contradictory or tautological proposition (indeed, they are generally false in this case). Thus we have also proven that the four conditions hold for all propositions. ■

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